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Supersymmetry, operator transformations and exactly solvable potentials

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Abstract. A large class of potentials can be solved algebraically by using supersymmetry and shape invariance. In this paper we apply operator transformations (f transformations) to these algebraically solvable problems to obtain a larger class of solvable potentials—the Natanzon class of potentials which are not shape invariant. The important condition (which is independent of supersymmetry) for finding new potentials from old ones using operator transformations is that the resulting Schrödinger equation has a potential which does not depend on the state. As a special case of the f transformation we rederive the previously known connection between the 3D harmonic oscillator, the hydrogen atom and the Morse potential. We also discuss the lack of commutivity of SUSY and the f transformations.

1. Introduction

In the pioneering work of Infeld and Hull [1] the conditions for the algebraic solution of the bound-state problem were presented. These conditions, namely factorisability and shape invariance, were later rediscovered in the context of supersymmetric quantum mechanics [2] by Gendenshtein and others [3, 4]. The conditions are met by a wide class of solvable potentials—Coulomb, harmonic oscillator, Morse, Eckart, Poschl-Teller, Rosen-Morse, etc. Recently we have shown how to extend these results to the calculation of the S matrix [5]. In the course of our studies we found that the Natanzon class [6] of solvable potentials were not directly solvable by the above approach since the general class was not shape invariant [7].

Related to the above work, but from a different perspective, there has been some recent work on the algebraic solution of potential problems using potential groups [8, 9]. There it was shown how an underlying potential group relating potentials of different strengths allows for an algebraic solution of the above potentials as well as the Natanzon potential.

In this paper we would like to show that, if we start off with a shape-invariant potential which is algebraically solvable, we can find new exactly solvable potentials which include the Natanzon class of potentials by considering operator transformations (f transformations) applied to the shape-invariant potentials. These operator transformations are a general method of obtaining new solvable potentials from already solvable potentials and do not depend on supersymmetry. We will show that these f transformations in general do not preserve shape invariance; nor do they take supersymmetric

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partner potentials into new supersymmetric partner potentials. The wavefunctions for the new potential are related to wavefunctions of the old potential whose parameters are a function of the energy eigenvalues of the new potential.

2. Review of SUSY, factorisation and shape invariance

In this section we review how the eigenspectrum and eigenstates of a class of one-dimensional Hamiltonians can be derived algebraically using SUSY and shape invariance. A more technical discussion of this problem not relying on supersymmetry is found in Infeld and Hull [1].

All Hamiltonians with a ground state Ψ_0 with energy E_0 can be factorised:

$$H_1\Psi_n^{(1)}=[-\mathrm{d}^2/\mathrm{d}r^2+V_1(r,a_1)]\Psi_n^{(1)}=(A_1^\dagger A_1+E_0^{(1)})\Psi_n^{(1)}=E_n^{(1)}\Psi_n^{(1)} \tag{2.1a}$$

where

$$A_1^\dagger=-\partial_r+W(r,a_1) \qquad A_1=\partial_r+W(r,a_1) \tag{2.1b}$$

$$V_1=W^2(r,a_1)-\partial_x W(r,a_1)+E_0^{(1)} \tag{2.1c}$$

$$\partial_r=\mathrm{d}/\mathrm{d}r. \tag{2.1d}$$

The a_1 are the parameters describing the potential, n labels the states, $n=0,1,\ldots,$ and $n=0$ denotes the ground state. The superpotential $W(r,a_1)$ is given by

$$W(r,a_1)=-\partial_r(\ln\Psi_0^{(1)}(r,a_1)). \tag{2.1e}$$

For simplicity let us set $E_0^{(1)}=0$. Then the partner Hamiltonian

$$H_2=A_1A_1^\dagger=-\mathrm{d}^2/\mathrm{d}r^2+V_2 \tag{2.2a}$$

$$V_2=W^2(r,a_1)+\partial_r W(r,a_1) \tag{2.2b}$$

gives the same spectrum as $H_1=A_1^\dagger A_1$ but with the ground state missing, i.e.

$$E_0^{(1)}=0 \qquad E_n^{(2)}=E_{n+1}^{(1)}. \tag{2.3}$$

In addition the eigenfunctions with the same energy are related:

$$\Psi_{n+1}^{(1)}(r)=(E_{n+1}^{(1)})^{-1/2}A^\dagger\Psi_n^{(2)}(r). \tag{2.4}$$

The degeneracy in the two spectra is due to a supersymmetry. We define the super Hamiltonian H and the supercharges Q and $Q^\dagger$:

$$H=\begin{bmatrix}H_1 & 0\\ 0 & H_2\end{bmatrix} \qquad Q=\begin{bmatrix}0 & 0\\ A_1 & 0\end{bmatrix} \qquad Q^\dagger=\begin{bmatrix}0 & A_1^\dagger\\ 0 & 0\end{bmatrix}. \tag{2.5}$$

These operators are the two-dimensional representation of the sl(1/1) superalgebra:

$$[Q,H]=[Q^\dagger,H]=0 \qquad \{Q,Q^\dagger\}=H \qquad \{Q,Q\}=\{Q^\dagger,Q^\dagger\}=0. \tag{2.6}$$

The fact that the supercharges commute with H gives rise to the energy degeneracy.

Clearly this process can be continued. Using factorisation H_2 can also be written as $A_2^\dagger A_2+\Delta E$, and the partner Hamiltonian of this can be constructed. This produces a ladder of potentials, V_n [1, 3, 4]. If the partner potentials are 'shape invariant' [1, 3] in that V_2 has the same functional form as V_1 but different parameters except for an additive constant:

$$V_2(r,a_1)=V_1(r,a_2)+C(a_1) \tag{2.7a}$$

then the full ladder is shape invariant:

$$V_n(r, a_1) = V_1(r, a_n) + E_{n-1}^{(1)} \tag{2.7b}$$

and the energy spectrum and eigenfunctions of the original potential can be determined algebraically:

$$E_n^{(1)} = \sum_{k=1}^{n} C(a_k) \tag{2.8a}$$

$$\Psi_n^{(1)}(r, a_1) = \prod_{k=1}^{n} A^\dagger(r, a_k)\Psi_0^{(1)}(r, a_{k+1}). \tag{2.8b}$$

3. Operator transformations (f transformations) and new potentials from old

We now ask the question whether by knowing the solution of one Schrödinger equation we can determine the solutions to other Schrödinger equations. For convenience, we will call our initial coordinates r and our final coordinates x. Although the use of operator transformations does not rely at all on supersymmetry, we will start with two Hamiltonians H_1 and H_2 given in (2.1) and (2.2) which are partners under the supersymmetry operation so that we can also study the question of whether these operator transformations commute with supersymmetry and whether they transform shape-invariant potentials into shape-invariant potentials.

Thus, starting with H_1 and H_2 we make a change of coordinates $r \to x$, defined by

$$f = \mathrm{d}x/\mathrm{d}r. \tag{3.1}$$

After the coordinate transformation, we introduce new operators and wavefunctions:

$$\bar{B} = f^{1/2}A^\dagger f^{-1/2} = -f\partial_x + \tfrac{1}{2}(\partial_x \ln f) + W \tag{3.2a}$$

$$B = f^{1/2}Af^{-1/2} = f\partial_x - \tfrac{1}{2}(\partial_x \ln f) + W \tag{3.2b}$$

$$\tilde{\Psi}_n^{(1)} = f^{1/2}\Psi_n^{(1)} = (E_n^{(1)})^{-1/2}\bar{B}\tilde{\Psi}_{n-1}^{(2)} \tag{3.2c}$$

$$\tilde{\Psi}_n^{(2)} = f^{1/2}\Psi_n^{(2)}. \tag{3.2d}$$

We now obtain the equations

$$f^{-2}(\bar{B}B - E_n^{(1)})\tilde{\Psi}_n^{(1)} = 0 \qquad f^{-2}(B\bar{B} - E_n^{(2)})\tilde{\Psi}_n^{(2)} = 0 \tag{3.3}$$

which are not yet in the form of an eigenvalue problem. To obtain a new eigenvalue problem we must add $\varepsilon_n\tilde{\Psi}_n$ to both sides of this equation [9]. We then find a new related pair of Schrödinger equations:

$$\tilde{H}_1\Psi_n^{(1)} = \varepsilon_n^{(1)}\Psi_n^{(1)} \qquad \tilde{H}_2\Psi_n^{(2)} = \varepsilon_n^{(2)}\Psi_n^{(2)} \tag{3.4a}$$

where

$$\tilde{H}_1 = \frac{1}{f^2}(\bar{B}B - E_n^{(1)}) + \varepsilon_n^{(1)} \tag{3.4b}$$

$$\tilde{H}_2 = \frac{1}{f^2}(B\bar{B} - E_n^{(2)}) + \varepsilon_n^{(2)}. \tag{3.4c}$$

In order for $\tilde{H}$ to correspond to a well defined potential it must not depend explicitly on ε. In general this requires that the parameters a that describe the strength of the superpotential $W(r, a)$ of H must now depend on ε. Thus every energy eigenvalue in the transformed Hamiltonian $\tilde{H}$ corresponds to a different set of parameters of the original shape-invariant potential. We call this combination of coordinate transformation and adding terms to the original equation an operator transformation. We can write the two related Schrödinger equations as follows:

$$[-\mathrm{d}^2/\mathrm{d}x^2+(1/f^2)\{\tfrac{1}{2}\partial_r^2(\ln f)-\tfrac{1}{4}[\partial_r(\ln f)]^2+W^2 \\ -\partial_r W-E_n^{(1)}\}+\varepsilon_n^{(1)}]\tilde{\Psi}_n^{(1)}=\varepsilon_n^{(1)}\tilde{\Psi}_n^{(1)} \tag{3.5a}$$

$$[-\mathrm{d}^2/\mathrm{d}x^2+(1/f^2)\{\tfrac{1}{2}\partial_r^2(\ln f)-\tfrac{1}{4}[\partial_r(\ln f)]^2+W^2 \\ +\partial_r W-E_n^{(2)}\}+\varepsilon_n^{(2)}]\tilde{\Psi}_n^{(2)}=\varepsilon_n^{(2)}\tilde{\Psi}_n^{(2)} \tag{3.5b}$$

or

$$[-\mathrm{d}^2/\mathrm{d}x^2+\tilde{V}^{(i)}]\Psi_n^{(i)}=\varepsilon_n^{(i)}\Psi_n^{(i)} \qquad i=1,2. \tag{3.5c}$$

If we assume that the transformation $f=\mathrm{d}x/\mathrm{d}r$ is energy independent then we obtain the following conditions for $\tilde{H}$ to be independent of ε:

$$V_1-E_n^{(1)}+\varepsilon_n^{(1)}f^2=G_1 \tag{3.6a}$$

$$V_2-E_n^{(2)}+\varepsilon_n^{(2)}f^2=G_2 \tag{3.6b}$$

where G is independent of ε_n and we have used (2.1c) and (2.2b).

In order to satisfy (3.6) several conditions must be true. First, the form of the transformation $f(r)$ is fixed—one requires that f^2 must have the same form as V apart from a constant:

$$f^2=V_1(r, b_1)+v \tag{3.7}$$

(here b_1 stands for the parameters controlling the shape of the potential V_1). Second, in order for $\tilde{V}$ to be state independent, the wavefunctions for each ε_n are related to the solutions of the original potential problem V with the parameters specifying V being a function of n. Third, from (3.6) and (3.7) we can determine ε_n in terms of E_n. These points will be made clear when we implement (3.6) and (3.7) for particular choices of f starting with some solvable shape-invariant potentials.

Next we want to address the question of whether these f transformations preserve shape invariance and whether they preserve supersymmetry. To do this we must first factorise $\tilde{H}_1$ and find its partner Hamiltonian $\tilde{H}_{ss}$:

$$\tilde{H}_1=\tilde{A}^\dagger\tilde{A}+\varepsilon_0^{(1)} \tag{3.8a}$$

where

$$\tilde{A}=\partial_x+\tilde{W} \tag{3.8b}$$

and the transformed superpotential is

$$\tilde{W}=W/f-\tfrac{1}{2}\partial_x \ln f. \tag{3.8c}$$

Note that

$$B=f\tilde{A}. \tag{3.8d}$$

The transformed potential is

$$\tilde{V}_1(x, a_1, f) = \frac{1}{f^2}\left(\frac{1}{2}\frac{d^2}{dr^2}(\ln f) - \tfrac{1}{4}(\partial_r \ln f)^2 + W^2 - \partial_r W\right). \tag{3.9}$$

The supersymmetric partner Hamiltonian is

$$\tilde{H}_{ss} = \tilde{A}\tilde{A}^\dagger + \varepsilon_0^{(1)} \tag{3.10}$$

with corresponding

$$V_{ss}(x, a_1, f) = \tilde{V}_1(x, a_2, f) + \frac{E_1^{(1)}}{f^2} - \frac{d^2}{dx^2}\ln f - 2(\partial_x \ln f)\frac{W}{f}. \tag{3.11}$$

From (3.11) we see that the transformed potential is not, in general, shape invariant. The condition for shape invariance is clearly that the sum of the last three terms on the right-hand side of (3.11) have the same structural form as $\tilde{V}_1$.

We can also compare the partner potential $\tilde{V}_{ss}$ with the f-transformed partner of V_1, V_2. We obtain

$$V_{ss}(x, a_1, f) = \tilde{V}_2(x, a_1, f) - \frac{d^2}{dx^2}\ln f - 2(\partial_x \ln f)\frac{W}{f} + \varepsilon_0^{(2)} - \varepsilon_0^{(1)}. \tag{3.12}$$

We see that f transforms, in general, do not commute with SUSY. The condition for the commutivity is that the sum of the last four terms on the right-hand side of (3.12) is zero.

4. Natanzon potentials as f transforms of the generalised Poschl–Teller potential

The generalised Poschl–Teller potential is given by

$$H_1 = A^\dagger A = -\frac{d^2}{dr^2} + \frac{\beta(\beta-1)}{\sinh^2 r} - \frac{\alpha(\alpha+1)}{\cosh^2 r} + (\alpha-\beta)^2 \tag{4.1}$$

where

$$W = \alpha \tanh r - \beta \coth r. \tag{4.2}$$

Its energy eigenvalues are

$$E_n^{(1)} = (\alpha-\beta)^2 - (\alpha-\beta-2n)^2 \tag{4.3}$$

where n are all non-negative integers less than $(\beta-\alpha)/2$. Its supersymmetric partner is

$$H_2 = AA^\dagger = -\frac{d^2}{dr^2} + \frac{\beta(\beta+1)}{\sinh^2 r} - \frac{\alpha(\alpha-1)}{\cosh^2 r} + (\alpha-\beta)^2 \tag{4.4}$$

with eigenenergies

$$E_n^{(2)} = (\alpha-\beta)^2 - (\alpha-\beta-2n-2)^2 \tag{4.5}$$

so that

$$E_n^{(2)} = E_{n+1}^{(1)}. \tag{4.6}$$

If we transform this pair of potentials the condition on the transformation function is given by (3.6) and (3.7):

$$f^2 = -\frac{A}{\cosh^2 r} + \frac{B}{\sinh^2 r} + C = \left(\frac{\mathrm{d}x}{\mathrm{d}r}\right)^2 \tag{4.7a}$$

where A, B, C are constants. The transformed potential becomes

$$\tilde{V}_1(x, \gamma, \delta) = \frac{1}{f^2}\left(\frac{\delta(\delta-1)}{\sinh^2 r} - \frac{\gamma(\gamma+1)}{\cosh^2 r} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(\ln f) - \tfrac{1}{4}(\partial_r \ln f)^2 + \sigma\right). \tag{4.7b}$$

Solving equation (3.6a) gives an implicit expression for the eigenvalues

$$[(\gamma+\tfrac{1}{2})^2 - A\varepsilon_n^{(1)}]^{1/2} - [(\delta-\tfrac{1}{2})^2 - B\varepsilon_n^{(1)}]^{1/2} - (\sigma - C\varepsilon_n^{(1)})^{1/2} = 2n+1 \tag{4.8a}$$

and determines the state-dependent α, β:

$$\alpha_n = [(\gamma+\tfrac{1}{2})^2 - A\varepsilon_n^{(1)}]^{1/2} - \tfrac{1}{2} \tag{4.8b}$$

$$\beta_n = [(\delta-\tfrac{1}{2})^2 - B\varepsilon_n^{(1)}]^{1/2} + \tfrac{1}{2}. \tag{4.8c}$$

The potentials (4.7) are the Natanzon potentials [6] which are the most general potentials which have hypergeometric functions as eigenfunctions. From (3.11) we see that these potentials are, in general, not shape invariant, but are the operator transform of a shape-invariant potential, a result proved very tediously in [7]. Hence, although the Natanzon potentials are not shape invariant, they are an operator transform of a shape-invariant potential.

5. Transformed harmonic oscillator

Starting from the 3D harmonic oscillator, we can use the results of the previous section to understand very simply the well known connection between the harmonic oscillator, Coulomb potential and Morse potential [10–13], as well as obtaining another class of Natanzon potentials. We start from the radial Schrödinger equation for the three-dimensional oscillator. The ground-state wavefunction (here β denotes the angular momentum)

$$\psi_0 = r^{\beta+1} \mathrm{e}^{-\alpha r^2/2} \tag{5.1}$$

yields $W = \alpha r - (\beta+1)/r$ so that the two partner Hamiltonians are

$$H_1 = A^+A = -\partial_r^2 + \beta(\beta+1)/r^2 + \alpha^2 r^2 - 2\alpha(\beta+\tfrac{3}{2}) \tag{5.2a}$$

$$H_2 = AA^+ = -\partial_r^2 + (\beta+1)(\beta+2)/r^2 + \alpha^2 r^2 - 2\alpha(\beta+\tfrac{1}{2}). \tag{5.2b}$$

For the tower of Hamiltonians we have the shape-invariance relationship:

$$H_n(\beta) = H_1(\beta+n-1) + 4\alpha(n-1). \tag{5.3}$$

This tells us from our previous discussion that, for H_1,

$$E_n = 4\alpha n. \tag{5.4}$$

Next we want the most general transformation that maps the harmonic oscillator into another energy-independent potential. That is, we need to have

$$W^2 - \partial_r W - 4\alpha n + \varepsilon_n f^2 = G. \tag{5.5}$$

We see that the most general form for f^2 and G consistent with this is

$$f^2 = A/r^2 + Br^2 + C \qquad G = D/r^2 + Er^2 + F. \tag{5.6}$$

For A–F to be energy independent we get from (3.6) the conditions (for H_1)

$$\beta(\beta+1) + \varepsilon_n A = D \qquad \alpha^2 + \varepsilon_n B = E \qquad -2\alpha(\beta+\tfrac{3}{2}) - 4\alpha n + \varepsilon_n C = F. \tag{5.7}$$

These conditions lead to the most general Natanzon potentials which are solvable by a confluent hypergeometric function. Again, using (3.11) we see these potentials are not shape invariant in general.

In order to make the transformation explicitly invertible there are two simple choices for f. One is

$$f(r) = r = \mathrm{d}x/\mathrm{d}r \qquad x = r^2/2. \tag{5.8}$$

This mapping is the well known mapping between the harmonic oscillator and the Coulomb potential [10]. Using (5.8) we find that

$$\tilde{V} = r^{-2}[-\tfrac{3}{4}r^{-2} + \alpha^2 r^2 - 2\alpha(\beta+\tfrac{3}{2}) + \beta(\beta+1)/r^2 - 4\alpha n + \varepsilon_n r^2] \tag{5.9}$$

$$= \tfrac{1}{4}x^{-2}[\beta(\beta+1) - \tfrac{3}{4}] - \tfrac{1}{2}x^{-1}[4\alpha n + 2\alpha(\beta+\tfrac{3}{2})] + (\alpha^2 + \varepsilon_n). \tag{5.10}$$

Thus we obtain the Coulomb potential with the identification

$$\beta = 2l + \tfrac{1}{2} \tag{5.11a}$$

$$4\alpha n + 2\alpha(\beta+\tfrac{3}{2}) = 2Ze^2 \qquad 4\alpha(l+n+1) = 2Ze^2 \tag{5.11b}$$

$$\alpha^2 + \varepsilon_n = \gamma. \tag{5.11c}$$

The mapping to the Coulomb potential shows us that, for each energy eigenvalue n and angular momentum l, a harmonic oscillator of different strength is needed:

$$\alpha(l, n) = Ze^2[2(l+n+1)]^{-1}. \tag{5.12}$$

This leads to the well known result that the energy eigenvalues of the Coulomb potential problem are given by

$$\varepsilon_n = \gamma - \tfrac{1}{4}Z^2e^4(l+1+n)^{-2}. \tag{5.13}$$

The Coulomb potential can also be cast into SUSY form starting from the ground-state wavefunction:

$$\Psi_0 = f^{1/2}\psi_0 = x^{l+1}\,\mathrm{e}^{-\alpha x} \tag{5.14}$$

where here $\gamma = \alpha(l, n=0)$.

The superpotential is

$$W_c = -(l+1)/x + \tfrac{1}{2}Ze^2/(l+1). \tag{5.15}$$

The partner Hamiltonians are

$$A_c = \mathrm{d}/\mathrm{d}x + W_c \tag{5.16}$$

$$H_1^c = -\mathrm{d}^2/\mathrm{d}x^2 + l(l+1)/x^2 - Ze^2/x + \tfrac{1}{4}Z^2e^4(l+1)^{-2} \tag{5.17}$$

$$H_2^c = -\mathrm{d}^2/\mathrm{d}x^2 + (l+1)(l+2)/x^2 - Ze^2/x + \tfrac{1}{4}Z^2e^4(l+1)^{-2}. \tag{5.18}$$

We see that the usual SUSY for the Coulomb potential relates potentials with the same Z but with angular momentum differing by one. On the other hand, the Hamiltonian we obtain by operator transforming the SUSY partner of the harmonic oscillator is

$$\tilde{H}_2 = -\mathrm{d}^2/\mathrm{d}x^2 + \tfrac{1}{4}x^{-2}[(\beta+1)(\beta+2) - \tfrac{3}{4}] - \tfrac{1}{2}x^{-1}[4\alpha n + 2\alpha(\beta+\tfrac{1}{2})] + (\alpha^2 + \varepsilon_n). \tag{5.19}$$

Using the parameter identification of (5.11), this corresponds to a Coulomb problem with angular momentum $l+\frac{1}{2}$ and charge $Z_+ = Z[1-\frac{1}{2}(l+1+n)^{-1}]$. Thus we see that only every other Hamiltonian in the infinite sequence of shape-invariant harmonic oscillator Hamiltonians gets mapped into an integer-valued angular momentum of the related Coulomb problem. The charges get changed as also noted by Haymaker and Rau [13]. Also note that the SUSY partner of the Coulomb potential is not the transformed partner of the oscillator problem. In order to solve the Coulomb problem one does not need to know anything about the transformed partner.

If we take the other special integrable case, where

$$dx/dr = f = 2/\delta r \qquad r = e^{(x\delta)/2} \tag{5.20}$$

we obtain the two related Schrödinger equations:

$$\tilde{H}_1 = -d^2/dx^2 + \varepsilon_n + \tfrac{1}{4}\delta^2(\beta+\tfrac{1}{2})^2 + \tfrac{1}{4}\alpha^2\delta^2 e^{2\delta x} - \varepsilon^{\delta x}\delta^2\alpha(n+\tfrac{1}{2}\beta+\tfrac{3}{4}) \tag{5.21a}$$

$$\tilde{H}_2 = -d^2/dx^2 + \varepsilon_n + \tfrac{1}{4}\delta^2(\beta+\tfrac{3}{2})^2 + \tfrac{1}{4}\alpha^2\delta^2 e^{2\delta x} - \varepsilon^{\delta x}\delta^2\alpha(n+\tfrac{1}{2}\beta+\tfrac{1}{4}). \tag{5.21b}$$

The Morse potential and its usual SUSY partner can be defined (apart from a constant) as follows.

Let $W_M(x) = A - B\,e^{\delta x}$. Then we have

$$\tilde{H}_M^{\pm} = -d^2/dx^2 + A^2 + B^2 e^{2\delta x} - 2B(A \pm \tfrac{1}{2}\delta)\,e^{\delta x}. \tag{5.22}$$

We eliminate the energy dependence in (5.21) by choosing

$$\gamma = \varepsilon_n + \tfrac{1}{4}\delta^2(\beta+\tfrac{1}{2})^2. \tag{5.23}$$

Comparing with the Morse potential we want to choose

$$B = \alpha\delta/2. \tag{5.24}$$

For $\tilde{H}_1$ we have the identification

$$n + \tfrac{1}{2}(\beta+\tfrac{3}{2}) = (A/\delta - \tfrac{1}{2}). \tag{5.25}$$

This leads to β being a function of n:

$$\beta + \tfrac{1}{2} = 2[A/\delta - (n+1)] \tag{5.26}$$

yielding the usual result

$$\varepsilon_n = \gamma - [A-(n+1)\delta]^2. \tag{5.27}$$

The potential obtained from the SUSY partner of the harmonic oscillator $\tilde{H}_2$ can be seen to have the parameter A given by $A-\delta/2$, whereas the SUSY partner $\tilde{H}_M^{+}$ has the parameter A given by $A+\delta$. This again shows that there is no relationship between these two partner potentials. These simple examples show the simplicity of the f transformation and show the lack of commutivity of the f transform with supersymmetry.

6. f transform of the 1D harmonic oscillator

If we start with the superpotential $W(r)$ corresponding to a shifted harmonic oscillator:

$$W(r) = \tfrac{1}{2}\omega r - b \qquad -\infty < r < \infty \tag{6.1}$$

then we have that

$$V_1(r)=\tfrac{1}{4}\omega^2r^2-b\omega r+b^2-\tfrac{1}{2}\omega. \tag{6.2}$$

The eigenvalues of this potential are

$$E_n=n\omega \tag{6.3}$$

and the eigenfunctions are shifted harmonic oscillator wavefunctions:

$$\Psi_n(\tilde{r})=N_nH_n(\omega^{1/2}\tilde{r})\exp(-\tfrac{1}{2}\omega\tilde{r}^2) \qquad \tilde{r}=r-2b/\omega. \tag{6.4}$$

From (3.7) we find that the most general form of the coordinate transformation f is given by

$$f^2=Ar^2+Br+C. \tag{6.5}$$

The simplest explicitly integrable case is $B=C=0$ so that

$$f=\mathrm{d}x/\mathrm{d}r=A^{1/2}r \tag{6.6}$$

or

$$r=(2x)^{1/2}/A^{1/4} \qquad x=\tfrac{1}{2}A^{1/2}r^2. \tag{6.7}$$

$\tilde{V}(x)$ has support in the interval $0<x<\infty$. In terms of r we have

$$\tilde{V}(r)=(Ar^2)^{-1}[-\tfrac{3}{4}r^{-2}+b^2-(n+\tfrac{1}{2})\omega-b\omega r+\tfrac{1}{4}r^2(\omega^2+4\varepsilon_nA)]. \tag{6.8}$$

For $\tilde{V}$ to be independent of n we clearly need

$$\varepsilon_n=-\omega^2/4A. \tag{6.9}$$

We can then write in terms of x:

$$\begin{aligned}\tilde{V}(x)&=-3/16x^2+(b^2-\omega/2-n\omega)/2xA^{1/2}-b\omega/2^{1/2}A^{3/4}x^{1/2}\\&=-3/16x^2+\alpha/x-\beta/x^{1/2}\end{aligned} \tag{6.10}$$

where α, β are independent of n.

The requirement that α and β are independent of n then tells us that, for each eigenvalue of $\tilde{V}$, we need to relate $\tilde{V}$ to a harmonic oscillator with different shape parameters ω and b. Specifically we find ω is the solution to the cubic equation:

$$(n+\tfrac{1}{2})\omega^3+2\alpha A^{1/2}\omega^2=2A^{3/2}\beta^2 \tag{6.11a}$$

and b satisfies

$$b=2^{1/2}A^{3/4}\beta/\omega. \tag{6.11b}$$

For the special case $\alpha=0$ this leads to

$$\begin{aligned}&\omega=[2A^{3/2}\beta^2/(n+\tfrac{1}{2})]^{1/3} \qquad b=(2A^{3/2}\beta^2)^{1/6}(n+\tfrac{1}{2})^{1/3}\\&\varepsilon_n=-\tfrac{1}{4}[2\beta^2/(n+\tfrac{1}{2})]^{2/3}.\end{aligned} \tag{6.12}$$

The wavefunctions $\tilde{\Psi}_n(x)$ are obtained from (3.2c). Using

$$f^{1/2}=A^{1/8}(2x)^{1/4} \tag{6.13}$$

we obtain

$$\tilde{\Psi}_n(x)=N_nx^{1/4}H_n(\omega^{1/2}\tilde{r})\exp(-\tfrac{1}{2}\omega\tilde{r}^2) \tag{6.14a}$$

where

$$\omega^{1/2}\tilde{r}=x^{1/2}[2\beta^2/(n+\tfrac{1}{2})]^{1/6}-(2n+1)^{1/2}. \tag{6.14b}$$

7. Conclusions

In this paper we have shown how to use operator transformations (f transformations) to obtain new solvable potentials from already known solvable potentials. We showed how starting from known shape-invariant potentials such as the Poschl–Teller potential and the harmonic oscillator potential we could obtain the wavefunctions and eigenvalues of a more general class of potentials—the Natanzon potentials. We also showed that these f transformation do not, in general, preserve shape invariance nor do they commute with supersymmetry operations.

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